

The modularity conjecture holds for linear idempotent varieties

Wolfram Bentz

Centro de Álgebra da Universidade de Lisboa
Av. Prof. Gama Pinto, 2
1649-003 Lisboa, Portugal, wfbentz@fc.ul.pt

Luís Sequeira

Centro de Álgebra da Universidade de Lisboa
Av. Prof. Gama Pinto, 2
1649-003 Lisboa, Portugal

&

Departamento de Matemática
Faculdade de Ciências, Universidade de Lisboa
1749-016 Lisboa, Portugal, lfsequeira@fc.ul.pt

December 24, 2012

Abstract

The “Modularity Conjecture” is the assertion that the join of two nonmodular varieties is nonmodular. We establish the veracity of this conjecture for the case of linear idempotent varieties. We also establish analogous results concerning n -permutability for some n , and the satisfaction of nontrivial congruence identities. Our theorems require a technical result about the equational theory of linear varieties, which might be of independent interest.

2010 *Mathematics Subject Classification.* 08B10, 08B05, 03C05.

Keywords: Interpretability lattice, congruence modularity, derivative, linear variety.

1 Introduction

The lattice \mathbf{L} of interpretability types was introduced in [6] and thoroughly studied in [5]. Maltsev conditions, which are associated with many important properties of varieties — such as permutability or distributivity of congruence lattices — correspond very nicely to filters of \mathbf{L} .

We assume the reader is familiar with the basic notions of interpretability of varieties and Maltsev conditions. For the relevant concepts, the reader is referred to [5] or [7].

The question of whether a given Maltsev condition may be implied by, or equivalent to, the conjunction of two weaker conditions translates directly into the question of whether the corresponding Maltsev filter fails to be prime, or indecomposable.

It was shown by Garcia and Taylor [5] that the filter of congruence distributive varieties is the proper intersection of larger filters. A similar result holds for the filter of all varieties with a near-unanimity operation [8].

On the other hand, [5] contains the following outstanding *modularity conjecture*:

Modularity Conjecture 1. *In L , the filter \mathcal{M} of congruence modular varieties is prime.*

In other words, the modularity conjecture states that the join of two varieties cannot have “Day terms” unless they already exist in one of them.

It was also conjectured in [5] that the filter of congruence permutable varieties is prime; this was proved by Steven Tschantz [11]. This “permutability conjecture” might be arguably easier than the modularity conjecture, and yet Tschantz’s proof is extremely difficult and remains unpublished.

Some work on the modularity conjecture was done in [7]. The focus there was on the form of Day terms in a potential counterexample, and it was shown that if one exists, the required Day terms must be rather involved, thus ruling out the possibility of an “easy” counterexample (see [9]).

In the present paper we take a different approach, focusing on varieties of a specific form.

We use the notion of *derivative* introduced by Dent, Kearnes and Szendrei in [3], and the results obtained there, to prove that the modularity conjecture holds when restricted to linear idempotent varieties (all the relevant definitions appear in the next section).

The main result of this paper is:

Theorem 1.1. *Let \mathcal{V}_1 and \mathcal{V}_2 be varieties axiomatized by linear idempotent identities. If $\mathcal{V}_1 \vee \mathcal{V}_2$ is congruence modular, then either \mathcal{V}_1 or \mathcal{V}_2 is congruence modular.*

In [3], the derivative is also used to give a characterization of those linear idempotent varieties which satisfy some nontrivial congruence identity. Using this characterization, here we are able to prove the following:

Theorem 1.2. *Let \mathcal{V}_1 and \mathcal{V}_2 be varieties axiomatized by linear idempotent identities. If $\mathcal{V}_1 \vee \mathcal{V}_2$ satisfies a nontrivial congruence identity, then either \mathcal{V}_1 or \mathcal{V}_2 satisfies a nontrivial congruence identity.*

Ralph Freese [4] created an ordered version of the derivative introduced in [3] and used it to establish, for the property of being n -permutable for some n ,

some results that are analogous to the ones established in [3] for modularity. Here we make use of the results in [4] to also establish the following result:

Theorem 1.3. *Let \mathcal{V}_1 and \mathcal{V}_2 be varieties axiomatized by linear idempotent identities. If $\mathcal{V}_1 \vee \mathcal{V}_2$ is n -permutable for some n , then either \mathcal{V}_1 or \mathcal{V}_2 is n -permutable for some n .*

Definitions of the above notions will be given in Section 2, while the results using the derivative and order derivative are shown in Sections 3 and 4, respectively. Section 5, which is essentially self-contained, gives the proof of a crucial property of linear varieties. This result might be of independent interest for the study of linearly generated equational systems.

2 Definitions and notations

Let Σ be a set of identities. We say that Σ is *idempotent* if, for every function symbol F appearing in Σ , it is the case that $\Sigma \models F(x, \dots, x) \approx x$. We say that Σ is *linear* if each term appearing in Σ has at most one function symbol.¹

We say that Σ is *inconsistent* if it can only be modeled in trivial varieties, i. e., if $\Sigma \models x \approx y$.

We'll say that a variety is (linear) idempotent whenever it is axiomatized by a (linear) idempotent set of identities.

Definition 2.1 ([3]). Let Σ be an idempotent set of identities and let F be a function symbol occurring in Σ .

- (i) We say that F is *weakly independent* of its i -th place if $\Sigma \models x \approx F(\mathbf{w})$ for a variable x and some sequence of not necessarily distinct variables \mathbf{w} , such that $w_i \neq x$.
- (ii) We say that F is *independent* of its i -th place if $\Sigma \models F(\mathbf{w}) \approx F(\mathbf{w}')$ where \mathbf{w}, \mathbf{w}' are two sequences of distinct variables, that are the same except at position i .

Example 1. Let $\Sigma = \{p(x, y, y) \approx x, p(y, y, x) \approx x\}$, the set of identities describing a Maltsev term. Then by the first identity p is weakly independent of its second and third positions; by the second identity, p is also weakly independent of its first position.

Definition 2.2 ([3]). Let Σ be an idempotent set of identities. The *derivative* Σ' is defined by augmenting Σ with the identities asserting that F is independent of its i -th place, for all F and i such that F is weakly independent of its i -th place.

Hence, the derivative Σ' can be seen as the result of strengthening every occurrence of weak independence in Σ to independence. When we have a variety \mathcal{V} axiomatized by a set Σ of idempotent identities, we will denote by \mathcal{V}' the

¹Linear terms are also called *depth 1* in [1], [2] and *simple* in [10].

variety axiomatized by Σ' . The derivative can be iterated in the obvious way. The n th derivative of Σ (or \mathcal{V}) will be denoted by $\Sigma^{(n)}$ (or $\mathcal{V}^{(n)}$).

Example 2. Let Σ be the same as in Example 1. Then Σ' will contain identities which state that p is independent of all its places:

$$\Sigma' = \Sigma \cup \{ p(u, y, z) \approx p(v, y, z), p(x, u, z) \approx p(x, v, z), p(x, y, u) \approx p(x, y, v) \}$$

It is easy to see that Σ' is inconsistent:

$$\Sigma' \models x \approx p(x, y, y) \approx p(y, y, y) \approx y$$

The notion of derivative was used in [3] to establish very nice results concerning congruence modularity, and the satisfaction of nontrivial congruence identities, which we will state and use in the next section. Dent, Kearnes and Szendrei also suggested in [3] that alternative notions of “derivative” might be developed, which could be applied in a similar fashion to other Maltsev properties, such as n -permutability. Ralph Freese ([4]) did just that, by defining the notion of *order derivative*:

Definition 2.3 ([4]). Let Σ be an idempotent set of equations. The *order derivative* of Σ , denoted by Σ^+ , is the augmentation of Σ by additional identities, in the following way: if $\Sigma \models x \approx F(\mathbf{w})$, for a tuple \mathbf{w} of not necessarily distinct variables, and an operation symbol F occurring in Σ , then Σ^+ will contain all identities of the form

$$x \approx F(\mathbf{w}')$$

where, for each i , w'_i is either x or w_i .

Example 3. Again, let Σ be as in Example 1. Then Σ^+ will include identities such as $x \approx p(x, x, y)$, $x \approx p(x, y, x)$ and $x \approx p(y, x, x)$ (and also identities that state that F is idempotent, but those add no information, as that already held in Σ). Clearly, Σ^+ is inconsistent, for

$$\Sigma^+ \models x \approx p(x, y, y) \approx y$$

When we have a variety \mathcal{V} axiomatized by a set Σ of identities, we will let \mathcal{V}^+ denote the variety axiomatized by Σ^+ . The n th order derivative of Σ (or of \mathcal{V}) will be denoted by $\Sigma^{+(n)}$ (or $\mathcal{V}^{+(n)}$).

3 Modularity and nontrivial congruence identities

Theorem 3.1 ([3], Thm. 3.2). *A variety \mathcal{V} is congruence modular if and only if \mathcal{V} realizes some set Σ of idempotent identities such that Σ' is inconsistent.*

It is noted in [3] that, while the fact of Σ' being inconsistent forces any variety realizing Σ to be congruence modular, the converse is not true in general. However, for *linear* idempotent varieties, a stronger result could be obtained. The following theorem is a slight reformulation of Theorem 5.1 of [3]:

Theorem 3.2. *Let \mathcal{V} be a linear idempotent variety. Then \mathcal{V} is congruence modular if and only if \mathcal{V}' is a trivial variety.*

Throughout this and the following section, Σ_1 and Σ_2 will always denote idempotent sets of identities, taken with disjoint sets of function symbols.

If $\mathcal{V}_1, \mathcal{V}_2$ are the varieties axiomatized by Σ_1, Σ_2 , respectively, then it is well known (see [5]) that $\mathcal{V}_1 \vee \mathcal{V}_2$ is exactly the variety axiomatized by $\Sigma_1 \cup \Sigma_2$.

Furthermore, it is clear that $(\Sigma_1 \cup \Sigma_2)' \supseteq \Sigma_1' \cup \Sigma_2'$. Corollary 3.4 below shows that in fact equality holds when Σ_1 and Σ_2 are *linear*. This result will come out as a consequence of Lemma 3.3. This lemma provides the leverage we need to establish all the results stated in the introduction. While the result stated in the lemma may seem intuitively obvious, its proof involves arguments about term manipulations that are quite technical in nature, and is presented in Section 5.

Lemma 3.3. *Let Σ_1 and Σ_2 be consistent sets of linear idempotent identities. Let F be an operation symbol occurring in Σ_i , where $i \in \{1, 2\}$. Consider a linear equation of the form $F(\mathbf{x}) \approx y$ over a variable set X . If $\Sigma_1 \cup \Sigma_2 \models F(\mathbf{x}) \approx y$, then $\Sigma_i \models F(\mathbf{x}) \approx y$.*

Corollary 3.4. *If Σ_1 and Σ_2 are sets of linear idempotent identities, then*

$$(\Sigma_1 \cup \Sigma_2)' = \Sigma_1' \cup \Sigma_2'$$

Proof. Immediate from the previous lemma. \square

Corollary 3.5. *Let $\mathcal{V}_1, \mathcal{V}_2$ be linear idempotent varieties. Then $(\mathcal{V}_1 \vee \mathcal{V}_2)' = \mathcal{V}_1' \vee \mathcal{V}_2'$.*

Corollary 3.6. *Let $\mathcal{V}_1, \mathcal{V}_2$ be linear idempotent varieties. Then for every $n \in \mathbb{N}$, $(\mathcal{V}_1 \vee \mathcal{V}_2)^{(n)} = \mathcal{V}_1^{(n)} \vee \mathcal{V}_2^{(n)}$.*

We also need the following fact from [5]:

Lemma 3.7 ([5]). *$\mathbf{1}$ is join-prime in L .*

Now we have all the tools we need to prove our main theorem.

Proof of Theorem 1.1. Let $\mathcal{V}_1, \mathcal{V}_2$ be linear idempotent varieties.

Suppose that \mathcal{V}_1 and \mathcal{V}_2 are not congruence modular. By Theorem 3.2, \mathcal{V}_1' and \mathcal{V}_2' are not trivial, i.e., they are different from $\mathbf{1}$ in L . Let $\mathcal{V} = \mathcal{V}_1 \vee \mathcal{V}_2$. By Corollary 3.5, $\mathcal{V}' = \mathcal{V}_1' \vee \mathcal{V}_2'$, and Lemma 3.7 guarantees that \mathcal{V}' is not trivial. Since \mathcal{V} is a linear idempotent variety, Theorem 3.2 gives us that \mathcal{V} is not congruence modular. \square

Theorem 3.2 shows that, for linear idempotent varieties, congruence modularity is equivalent to the derivative being inconsistent. The following result relates the satisfaction of some nontrivial congruence identity to the inconsistency of some iteration of the derivative.

Theorem 3.8 ([3]). *Let \mathcal{V} be a linear idempotent variety. Then \mathcal{V} satisfies some nontrivial congruence identity if and only if for some n , $\mathcal{V}^{(n)}$ is trivial.*

Proof of Theorem 1.2. Let $\mathcal{V}_1, \mathcal{V}_2$ be linear idempotent varieties, and suppose that $\mathcal{V}_1 \vee \mathcal{V}_2$ satisfies some nontrivial congruence identity. By Theorem 3.8, there is some natural number n such that $(\mathcal{V}_1 \vee \mathcal{V}_2)^{(n)}$ is trivial. Hence, by Corollary 3.6 and Lemma 3.7, either $\mathcal{V}_1^{(n)}$ or $\mathcal{V}_2^{(n)}$ is trivial. Again, by Theorem 3.8, we have that either \mathcal{V}_1 or \mathcal{V}_2 satisfies a nontrivial congruence identity. \square

4 n -permutability

In this section, we take care of the proof of Theorem 1.3, which will follow along the very same lines as those of Theorems 1.1 and 1.2. The first result is an analog, for the order derivative, of Corollary 3.4:

Lemma 4.1. *If Σ_1 and Σ_2 are sets of linear idempotent identities, then*

$$(\Sigma_1 \cup \Sigma_2)^+ = \Sigma_1^+ \cup \Sigma_2^+$$

Proof. Again, this is an immediate consequence of Lemma 3.3. \square

Corollary 4.2. *Let $\mathcal{V}_1, \mathcal{V}_2$ be linear idempotent varieties. Then for every $n \in \mathbb{N}$, $(\mathcal{V}_1 \vee \mathcal{V}_2)^{+(n)} = \mathcal{V}_1^{+(n)} \vee \mathcal{V}_2^{+(n)}$.*

The following result is part of Theorem 7 of [4].

Theorem 4.3. *Let Σ be a set of linear idempotent identities. Then the variety axiomatized by Σ is n -permutable for some n if and only if some iterated order derivative of Σ is inconsistent.*

Proof of Theorem 1.3. The result follows just as in the proof of Theorem 1.2, using the order derivative instead of the derivative. \square

5 Identities in linear varieties

In this section we prove the important technical Lemma 3.3, on which the proofs of Theorems 1.1, 1.2 and 1.3 were based. The proof uses arguments over rewriting sequences in a fashion similar to [1] and [2].

In the following we will introduce terminology for this task. Note that some of our notations are variants (usually generalizations) of established meanings. Also, some concepts have complicated formal descriptions but are easy to grasp informally; we will occasionally stick with an informal term whose meaning should be clear in order to avoid excessive notation.

For a given signature, consider a set Σ of linear equalities over a set V and the set $T(X)$ of Σ -terms over a set X , which we may consider to be infinite. For simplicity we will use x, y, z to refer to elements of X and v, w to refer to elements of V (occasionally with subscripts).

Given a term $t \in T(X)$, an *occurrence* s in t is a path in the syntactic tree of t together with the subterm of t corresponding to the subtree reached by the path. We will refer to the path as the *position* of the occurrence and use \bar{s} to denote the corresponding subterm. Note that a position can be identified with a (potentially empty) finite list of integers. Given an occurrence s in an occurrence t and another occurrence s' in an occurrence t' , we can talk of the occurrence that is in the same relative position towards s' as s is to t by concatenating the position of s' in t' with the position of s in t ; this requires that the resulting position is “compatible” with t' in an obvious way.

We will extend structural concepts from terms onto occurrences. For example, if we say that an occurrence t has the form $f(s_1, \dots, s_n)$ for some occurrences s_i , this will mean that the terms \bar{s}_i satisfy $\bar{t} = f(\bar{s}_1, \dots, \bar{s}_n)$ and that the position of s_i is the i -th child of t . If we introduce an occurrence t without positional context, its position should be taken as the root of the corresponding term \bar{t} .

A *derivation* in Σ is a sequence t_0, t_1, \dots, t_n of occurrences, such that t_i is obtained from t_{i-1} by one rewriting step with an equality $\epsilon_i \in \Sigma \cup \Sigma^\delta$. Existence of a derivation is clearly equivalent to $\Sigma \models t_0 \approx t_n$. We additionally require that a derivation is enriched with enough “syntactical information” to completely reconstruct the rewriting procedure. Concretely, for each pair (t_{i-1}, t_i) an explicit ϵ_i is given together with two occurrences s in t_{i-1} and s' in t_i such that t_i is obtained from t_{i-1} by rewriting \bar{s} into \bar{s}' using the equality ϵ_i . We will moreover adopt the convention that if $t \approx t'$ is a rewriting step using the equality $u \approx u'$, then u matches up with \bar{s} and u' with \bar{s}' . We will also consider reverse pairs (t_i, t_{i-1}) to be single rewriting steps of the derivation, in this case we use ϵ_i^δ as the corresponding equality.

Example 4. In the following examples, the relevant equality is applied to the occurrences specified by underlining.

1. Let $t = f(g(h(x), h(x), x))$, $t' = f(h(x))$, then t' is obtained from t by rewriting with the equation $g(v, v, w) \approx v$. Note that t' could have also been obtained through the equation $g(v_1, v_2, v_3) \approx v_1$, and that t is obtained from t' by rewriting with $v \approx g(v, v, w)$.
2. $t' = f(y, x)$ is obtained from $f(x, y)$ by the equation $f(u, v) \approx f(v, u)$. It could have also been obtained by the equation $f(v, u) \approx f(u, v)$.
3. $t' = f(f(f(x)))$ is obtained from $f(f(x))$ with the equation $f(v) \approx f(f(v))$, using the underlined occurrences. It could have also been obtained with the same equation applied to the occurrences given by $t = \underline{f(f(x))}$, $t' = \underline{f(f(f(x)))}$.

The syntactic information contained with each derivation will ensure that various constructions below will be well-defined. Now consider a fixed derivation $t = t_0, t_1, \dots, t_n = t'$. By an *occurrence of the derivation*, we mean an index $i \in \{0, \dots, n\}$ together with an occurrence s of t_i . Once again, let $\bar{s} \in T(X)$ denote the underlying term of s . The occurrences of a derivation are naturally

ordered by inclusion; we will denote this order by “ \leq ”. Note that occurrences from different t_i are always incomparable under \leq .

We will next define a quasi-order \succeq on the occurrences of a derivation in terms of a generating relation \succeq' . Let $u \leq t_i =: t$ and let t' be one of t_{i-1}, t_{i+1} , such that t' is obtained from t by rewriting with the equation ϵ . Let $s \leq t, s' \leq t'$ be the occurrences involved in the rewriting step. We will define pairs of the form $(u, _)$ in \succeq' according to particulars of the rewriting step in relation to u , by distinguishing several cases. For our definition to be well-defined, we need that all positions are actually valid in their containing occurrence; the routine verification of this is left to the reader.

Definition 5.1. With notation as above, we define pairs in \succeq' as follows:

1. If $u \not\leq s$ and $s \not\leq u$, let $u' \leq t'$ be the occurrence that is in the same position in t' as u is in t . In this case we set $u \succeq' u'$. Note that $\bar{u} = \bar{u}'$.
2. If $s < u$, let $u' \leq t'$ be the occurrence that is in the same position in t' as u is in t ; set $u \succeq' u'$. Note that \bar{u}' can be obtained from \bar{u} by rewriting with ϵ .
3. This case covers the situation that either $u < s$ (for arbitrary ϵ) or that $u = s$ and ϵ is of the form $v \approx f(v_1, \dots, v_n)$ or $v \approx v$ for some $v, v_i \in V$. As ϵ is linear, there is a unique $w \in V$ appearing in the left hand side of ϵ and a unique occurrence $p, u \leq p \leq s$, such that the rewriting step matches p with w (p is either equal to s , if ϵ has the form $v \approx f(v_1, \dots, v_n)$, in which case $v = w$, or is “one level below” s , in the other cases).

For each occurrence p_i^* with $\bar{p}_i^* = w$ in ϵ , let p_i be the occurrence in the same position in s or s' , as p_i^* is in the left or right hand side of ϵ , respectively. All of these occurrences have the same underlying term as p (note that p itself is one of the p_i ; it is possible that p is the only such occurrence). Let u_i be the occurrences that are in the same position in p_i as u is in p , all of which have the same underlying term as u . We set $u \succeq' u_i$ for each such u_i .

4. Let $u = s$ and ϵ have the form $f(v_1, \dots, v_i) = g(w_1, \dots, w_j)$ or $f(v_1, \dots, v_i) = v$. We set $u \succeq' s'$.

We define \succeq' to be the smallest set obtained by the above rules, as (t, t') run through all pairs of the form (t_i, t_{i-1}) and (t_i, t_{i+1}) , and u runs through all occurrences with $u \leq t_i$. We let \succeq be the reflexive and transitive closure of \succeq' .

Example 5. Below, we will give various examples of occurrences t , rewriting equation ϵ , and rewritten occurrence t' . As in Example 4, we will indicate syntactical information by underlining. An overbrace will indicate the occurrence corresponding to u while an underbrace will indicate all occurrences u' with $u \succeq' u'$. Our numbering corresponds to that in Definition 5.1.

1. $f(\overbrace{g(x), f(y)}; f(v) \approx v; \underbrace{f(g(x), y)})$

2. $f(\overbrace{g(x, f(y))}); f(v) \approx v; \underbrace{f(g(x, y))}$
3. (a) $\frac{f(\overbrace{g(h(x, y), h(x, y), h(x, y))}); g(v, v, w) \approx h(v, w); f(\overbrace{h(h(x, y), h(x, y))})}{\underbrace{f(\overbrace{g(x, y)}); v \approx h(v, v, w); f(\overbrace{h(g(x, y), g(x, y), g(x, y))})}}$
4. $f(\overbrace{g(h(x, y), h(x, y), h(x, y))}); g(v, v, w) \approx h(v, w); \underbrace{f(h(h(x, y), h(x, y)))}$

The following lemma is obvious from the definition of \succeq .

Lemma 5.2. *With notation as above, if $s \succeq s'$ then $\Sigma \models \bar{s} \approx \bar{s}'$.*

We now use the above definitions in the situation of Lemma 3.3 with $\Sigma := \Sigma_1 \cup \Sigma_2$. Without loss of generality, assume that F occurs in Σ_1 and that there is an identity

$$F(\mathbf{x}) \approx y \quad (Id)$$

such that $\Sigma_1 \cup \Sigma_2 \models (Id)$. Fix a derivation of this identity, t_0, \dots, t_n , with $F(\mathbf{x}) = \bar{t}_0$, $\bar{t}_n = y$, corresponding equalities $\epsilon_i \in \Sigma \cup \Sigma^\delta$, and related occurrences $s_i \leq t_{i-1}$, $s'_i \leq t_i$.

Claim 5.3. *With notation as above, we have that $t_0 \succeq t_n$.*

Let T be the set of all occurrences u in the derivation such that $t_0 \succeq u$. Pick a variable $z \in X$ that does not occur in the derivation. For each t_i we will define an occurrence p_i as follows. Consider each occurrence such that $u \leq t$ for some $t \in T$ and that the underlying term of u is a variable from the set X . Now let \bar{p}_i be the term that it obtained from t_i by replacing the variable of each such u with z .

Lemma 5.4. *p_0, \dots, p_n is once again a Σ -valid derivation, with the same ϵ_i and corresponding syntactical information.*

Proof. Consider the step from t_{i-1} to t_i , so that $\epsilon := \epsilon_i$ rewrites the occurrence $s := s_i \leq t_{i-1}$ into $s' := s'_i \leq t_i$. Let q denote the occurrence that is in the same position in p_{i-1} as s is in t_{i-1} , and let q' denote the occurrence that is in the same position in p_i as s' is in t_i .

We first show that \bar{q} is an instance of the left hand side of ϵ . This is trivial of the left hand side of ϵ is a variable. So assume that the left hand side of ϵ has the form $f(\mathbf{v})$. Let $v_j = v_k$ for some $j \neq k$. As \bar{s} is an instance of $f(\mathbf{v})$, we have that $s = f(\mathbf{d})$, for some occurrences satisfying $\bar{d}_j = \bar{d}_k$. As the definition of p_{i-1} only affects variables, $q = f(\mathbf{e})$, where each e_l can be obtained from the corresponding d_l by changing some variables to z . In order to show that q is an instance of $f(\mathbf{v})$, we have to show that this happens for the same occurrences

of variables in both d_j and d_k . We will go through the various cases in which a variable can switch.

If some occurrence $o \geq s$ is in T , then all variables in e_j and e_k are z and we have $\bar{e}_j = \bar{e}_k$, as needed. If any occurrence $o_j \leq d_j$ is in T , then the occurrence $o_k \leq d_k$ that is in the same position in d_k as o_j is in d_j belongs to T as well, by Case 2, and vice versa, by the symmetry in Case 2. Hence once again $\bar{e}_j = \bar{e}_k$ and we can conclude that q is an instance of $f(\mathbf{v})$.

It remains to show that the result of rewriting p_{i-1} with ϵ applied to occurrence q is p_i . It once again suffices to consider occurrences whose underlying terms are variables. So let $u \leq t_{i-1}$ be such an occurrence. Suppose that $u \not\leq s$. Then there is a u' in the corresponding position in t_i with $\bar{u} = \bar{u}'$. Now suppose that there is some occurrence $o \in T$ with $o \geq p$. Then o is either incomparable with s or $o > s$. In either case there is an occurrence $o' \in T$ in the same position in t_i as o is in t_{i-1} , by Case 1 or Case 3 of the definition of \succeq' . The same argument holds in reverse if we start with the assumption that u' is contained in an occurrence $o' \in T$ (recall that our definition of \succeq' also included the reverse rewriting operations from t_i to t_{i-1}). Hence u is contained in an occurrence from T if and only if u' is. Hence, when transforming from (t_{i-1}, t_i) to (p_{i-1}, p_i) , u changes its term to z if and only if u' changes. It follows that p_{i-1} and p_i are identical outside of the rewriting effected occurrences q and q' , as required.

Now let v be a variable appearing in the equality ϵ . Pick any occurrences $a \neq a'$ that correspond to v during the rewriting process. Then $\bar{a} = \bar{a}'$. In the case that $a, a' \leq s$, we have already seen that variables switch to z in exactly the same positions in a and a' when showing that q is an instance of the left hand side of ϵ , and the same result follows by symmetry if $a, a' \leq s'$. Hence assume w.l.o.g. that $a \leq s$, $a' \leq s'$.

Let $u \leq a$ with \bar{u} being a variable and u' be the occurrence that is in the same position in a' as u is in a . Now assume that there is an occurrence $o \in T$ with $o \geq u$. If $o \geq s$, then by Case 2, s' and hence a' and u' is also contained in an occurrence $o' \in T$. If $o = s$ and ϵ has a form as in Case 4, then by that case $s' \in T$, with $u' \leq s'$. If $o = s$ and ϵ has the form " $v \approx f(\mathbf{w})$ " or " $v \approx v$ ", then $a = s = o \in T$ and hence $u' \leq a' \in T$ by Case 3. If $u \leq o < s$, then $o \leq a$, and, also by Case 3, there is an $o' \in T$ with $u' \leq o' \leq a'$, namely the one that is in the same relative position in a' as o is in a . In all cases, if u is contained in an occurrence from T then so is u' , and by symmetry, the converse holds. As above $\bar{b} = \bar{b}'$ follows, where b, b' are the occurrences in p_{i-1}, p_i that correspond to a, a' , and hence are also matched to the same variable v in ϵ . Hence the underlying terms of any two occurrences matched to same variable in ϵ agree. As p_{i-1} is unchanged from p_i outside of the rewriting area, it follows that p_i can be obtained from p_{i-1} by rewriting with ϵ . Hence p_0, \dots, p_n is a Σ -valid derivation. \square

We are now able to prove Claim 5.3. By Lemma 5.4, $\Sigma \models \bar{p}_0 \approx \bar{p}_n$. As $p_0 \in T$ by definition, we have $\bar{p}_0 = F(z, \dots, z)$. If $p_0 \not\leq p_n$, i.e. $p_n \notin T$ we would have $\Sigma \models F(z, \dots, z) \approx y$ with $z \neq y$, which implies inconsistency. The claim

follows by contradiction from Lemma 3.7.

By the definition of T and \succeq there are occurrences $q_0 \succeq' q_1 \succeq' \dots \succeq' q_k$ with $q_0 = t_0$ and $\bar{q}_k = \bar{t}_n$. By potentially shortening the sequence, we may assume that $\bar{q}_i \neq y$ for $i < k$. It is easy to see by induction that for all $q_i \neq r_k$, \bar{q}_i has the form $f_i(\dots)$, where all f_i are function symbols from the signature of Σ_1 .

Now define a sequence of occurrences r_0, r_1, \dots, r_k by setting $\bar{r}_i = \bar{q}_i$ for all i .

Lemma 5.5. *The sequence r_0, r_1, \dots, r_k can be extended to a derivation over $\Sigma' = \Sigma \cup \{v \approx v\}$ by suitable syntactical information. Moreover, this can be done while avoiding any rewriting step using an equation of the form $v \approx f(\mathbf{w})$ in a way in which the left hand side variable v is matched to any r_i .*

Proof. Let $0 \leq i < k$. If $r_i \succeq' r_{i+1}$ follows from the rules of either Case 1 or Case 3, then $\bar{r}_i = \bar{r}_{i+1}$, and this is a rewriting step with the trivial equation $v \approx v$.

Assume next that $r_i \succeq' r_{i+1}$ follows by Case 2 or Case 4. Then $r_i \leq t$, $r_{i+1} \leq t'$ where $t = t_j$ for some j and t' is either t_{j-1} or t_{j+1} . Let t' be obtained from t by rewriting $s \leq t$ into $s' \leq t'$ using ϵ .

In Case 4, ϵ has the form $f(v_1, \dots, v_i) \approx g(w_1, \dots, w_j)$ or $f(v_1, \dots, v_i) \approx w$, $s = r_i$ and $s' = r_{i+1}$ and hence \bar{r}_{i+1} is obtained from \bar{r}_i by rewriting with ϵ , applied to the entire term, as required.

Finally, let the situation be as in Case 2. Then $s < r_i$, $s' < r_{i+1}$, and hence \bar{r}_{i+1} is obtained from \bar{r}_i by rewriting a strict smaller occurrence than r_i with ϵ .

This proves the first assertion. Now, in Cases 1, 3, and 4, the equality connecting r_i and r_{i+1} does not have the form $v \approx f(\mathbf{w})$, while in Case 2, the equality is applied to a strict smaller occurrence than r_i . The second assertion follows. \square

Now let R be the collection of all occurrences appearing in the derivation r_0, \dots, r_k . Choose a function φ from R to X satisfying

1. $\varphi(r) = \varphi(r')$ if and only if $\Sigma \models \bar{r} \approx \bar{r}'$.
2. $\Sigma \models \bar{r} \approx x$ implies $\varphi(r) = x$ for all $x \in X$.

As X is infinite such a function exists. We now define another sequence of occurrences. For $0 \leq i < k$, if r_i has the form $f(d_1, \dots, d_n)$ set $u_i := f(\varphi(d_1), \dots, \varphi(d_n))$. In addition, set $u_k = r_k$.

Lemma 5.6. *The sequence u_1, \dots, u_k can be extended to a derivation using only equations from $\Sigma_1 \cup \{v \approx v\}$.*

Proof. For $0 \leq i < k$, consider the rewriting step from r_i to r_{i+1} by the equation ϵ . If this rewriting step involves rewriting a strictly smaller occurrence than r_i , or if it is trivial, then $\bar{u}_i = \bar{u}_{i+1}$ by our definition of φ and so u_{i+1} follows from u_i by the equation $v \approx v$. Otherwise, \bar{r}_i must be an instance of the left hand side of ϵ . By Lemma 5.5 ϵ cannot have the form $v \approx f(\mathbf{w})$, and hence must look like $f(v_1, \dots, v_i) \approx g(w_1, \dots, w_j)$ or $f(v_1, \dots, v_i) \approx w$. As mentioned above, r_i , and

hence u_i has the form $f_i(\dots)$, with f_i being from the signature of Σ_1 . It follows that $\epsilon \in \Sigma_1$. Moreover, as ϵ is linear, \bar{u}_i is an instance of ϵ and is rewritten into \bar{u}_{i+1} by ϵ , once again by our definition of φ . The result follows. \square

We are now ready to show Lemma 3.3.

Proof. The previous lemma shows that $\Sigma_1 \models \bar{u}_0 \approx \bar{u}_k$. But $F(\mathbf{x}) = \bar{t}_0 = \bar{r}_0 = \bar{u}_0$, where the last equality follows from the second property of φ . In addition $\bar{u}_k = \bar{r}_k = y$ and hence $\Sigma_1 \models F(\mathbf{x}) \approx y$. The result follows. \square

Concluding remarks

Our results provide further evidence in support of the modularity conjecture. Theorem 1.1 shows that the conjecture holds in the special case of linear idempotent varieties. Our methods rely heavily on the properties of linear varieties, indicating that the general hypothesis is unlikely to be solved using this line of inquiry.

Our results about linear varieties, in particular Lemma 3.3, could prove useful in studying other properties of linear systems, potentially far removed from the topics of this paper.

Acknowledgment

The first author has received funding from the European Union Seventh Framework Programme (FP7/2007-2013) under grant agreement no. PCOFUND-GA-2009-246542 and from the Foundation for Science and Technology of Portugal.

The second author acknowledges support from FCT under Project PEst-OE/MAT/UI0143/2011.

References

- [1] Bentz, W. *A characterization of Hausdorff separation for a special class of varieties*, Algebra Universalis 55 (2006), 259–276
- [2] Bentz, W. *A Characterization of T_3 Separation for a Special Class of Varieties*, Algebra Universalis 56 (2007), 399–410
- [3] Dent, A.; Kearnes, K.; Szendrei, A. *An easy test for congruence modularity*, Algebra Universalis 67 (2012), 375–392
- [4] Freese, R. *Notes on congruence n -permutability and semidistributivity*, manuscript
- [5] Garcia, O.; Taylor, W. *The Lattice of Interpretability Types of Varieties*, Memoirs of the American Mathematical Society, Vol. 50, Number 305, 1984

- [6] Neumann, W. D. *On Mal'cev conditions*, J. Aus. Math. Soc. **17** (1974), 376–384
- [7] Sequeira, L. *Maltsev Filters*, Ph. D. Thesis, Universidade de Lisboa, 2001
<http://www.ptmat.fc.ul.pt/~lsequeir/math/tese.pdf>
- [8] Sequeira, L. *Near-unanimity is decomposable*, Algebra Universalis **50** (2003), 157–164
- [9] Sequeira, L. *On the modularity conjecture*, Algebra Universalis **55** (2006), 495–508
- [10] Taylor, W. *Simple equations on real intervals*, Algebra Universalis **61** (2009), 213–226
- [11] Tschantz, S. *Congruence permutability is join prime*, unpublished